非厄米量子传感器的
终极精度极限

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Classical Fisher information

Fisher information *for a specific measurement POVM*

$$\mathcal{I}_\lambda = \sum_\xi P(\xi|\lambda)\left(\frac{\partial \ln P(\xi|\lambda)}{\partial\lambda}\right)^2, \qquad P(\xi|\lambda) = \mathrm{Tr}(\Pi\rho_\lambda).$$

Cramer-Rao bound

$$\delta\lambda \geq \frac{1}{\sqrt{\nu\mathcal{I}_\lambda}},$$

ν is the number of repetition times

Quantum fisher information (QFI)

QFI *optimized over all possible POVM measurements*

$$\mathcal{I}_\lambda \leq F_\lambda.$$

$$F_\lambda = \mathrm{Tr}[\rho_\lambda \mathcal{L}^2], \qquad \frac{\partial \rho_\lambda}{\partial \lambda} = \frac{1}{2}(\mathcal{L}\rho_\lambda + \rho_\lambda \mathcal{L}).$$

symmetric logarithmic derivative

Quantum Cramer-Rao bound （unbiased estimation）
Closedly related to the Heisenberg uncertainty relations

$$\delta\lambda \geq \frac{1}{\sqrt{\nu F_\lambda}}.$$

Criticality-based quantum metrology

$$\hat{H} = \hat{H}_0 + \lambda \hat{H}_1,$$

time-independent Hamiltonian

Seminorm

$$G^{-1/2} \geq \frac{1}{t\|\hat{H}_1\|}.$$

$$\|H\| = M_H - m_H,$$

maximum (minimum) eigenvalue of H.

Fisher information is less or equal than t^2*seminorm^2

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At the Limits of Criticality-Based Quantum Metrology: Apparent Super-Heisenberg Scaling Revisited

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Exceptional point (EP)

$$H = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}, \qquad \alpha \in \mathbb{C},$$

coalescence of both eigenvalues and eigenstates，空间维数减少

$$E_{\pm} = \pm\sqrt{\alpha}$$

$$\psi_{R,\pm} = \begin{pmatrix} \pm\sqrt{\alpha} \\ 1 \end{pmatrix}, \qquad \psi_{L,\pm} = \begin{pmatrix} 1 & \pm\sqrt{\alpha} \end{pmatrix}.$$

$$\frac{\partial E_+}{\partial \alpha} = \frac{1}{2\sqrt{\alpha}}$$

$$|\partial_\alpha E(\alpha)| \to \infty \text{ as } \alpha \to 0,$$

EP sensor proposal

[19] J. Wiersig, Enhancing the Sensitivity of Frequency and Energy Splitting Detection by Using Exceptional Points: Application to Microcavity Sensors for Single-Particle Detection, Phys. Rev. Lett. **112**, 203901 (2014).

EP sensor experiment

[20] H. Hodaei, A. U. Hassan, S. Wittek, H. Garcia-Gracia, R. El-Ganainy, D. N. Christodoulides, and M. Khajavikhan, Enhanced sensitivity at higher-order exceptional points, Nature (London) **548**, 187 (2017).

[21] W. Chen, S. K. Özdemir, G. Zhao, J. Wiersig, and L. Yang, Exceptional points enhance sensing in an optical micro-cavity, Nature (London) **548**, 192 (2017).

Time-dependent unitary parameter encoding

$$\rho_\lambda(t) = U_\lambda(0 \to t)\rho_0 U_\lambda^\dagger(0 \to t), \qquad U_\lambda(0 \to t) = \mathcal{T}e^{-i\int_0^t \hat{H}_\lambda(s)ds},$$

$\hat{H}_\lambda(t)$ *is the parameter encoding generator*

Convexity(Tu) of QFI，the optimal input state is always a pure state

$$\rho(0) \ = \ |\Psi_0\rangle\langle\Psi_0|$$

QFI

$$\begin{aligned} F_\lambda(t) &= 4(\langle\Psi_0|h_\lambda^2(t)|\Psi_0\rangle - |\langle\Psi_0|h_\lambda(t)|\Psi_0\rangle|^2) \\ &\equiv 4\mathrm{Var}[h_\lambda(t)]|_{|\Psi_0\rangle}, \end{aligned}$$

Local generator

$$h_\lambda(t) = iU_\lambda^\dagger(0 \to t)\frac{\partial}{\partial\lambda}U_\lambda(0 \to t),$$

Hermitian Local generator

$$h_\lambda(t) = iU_\lambda^\dagger(0 \to t)\frac{\partial}{\partial\lambda}U_\lambda(0 \to t),$$

$$i\partial U_\lambda/\partial t = H_\lambda U_\lambda,$$

$$i\frac{\partial U^\dagger}{\partial t} = -U^\dagger H$$

$$H = i\partial_t U U^\dagger = -iU\partial_t U^\dagger.$$

与哈密顿的类似性

Time Derivative of Local Generator

$$\frac{\partial h_\lambda}{\partial t} = U_\lambda^\dagger(0 \to t)\frac{\partial H_\lambda(t)}{\partial \lambda}U_\lambda(0 \to t),$$

$$\begin{aligned}\frac{\partial h_\lambda}{\partial t} &= -U^\dagger H\frac{\partial U}{\partial \lambda} + U^\dagger\frac{\partial\left(HU\right)}{\partial \lambda}\\ &= -U^\dagger H\frac{\partial U}{\partial \lambda} + U^\dagger H\frac{\partial U}{\partial \lambda} + U^\dagger\frac{\partial H}{\partial \lambda}U\\ &= U^\dagger\frac{\partial H}{\partial \lambda}U\end{aligned}$$

Form of the Local Generator

$$h_\lambda(t) = \int_0^t U_\lambda^\dagger(0 \to s)\frac{\partial H_\lambda(s)}{\partial \lambda}U_\lambda(0 \to s)ds.$$

Seminorm

$$\mathrm{Var}[\hat{A}]|_{|\Psi\rangle} \leq ||\hat{A}||^2/4.$$

$$||\hat{A}|| = M_A - m_A$$

Variance is bounded by the seminorm

$$F_\lambda(t) = 4(\langle\Psi_0|h_\lambda^2(t)|\Psi_0\rangle - |\langle\Psi_0|h_\lambda(t)|\Psi_0\rangle|^2) \equiv 4\mathrm{Var}[h_\lambda(t)]|_{|\Psi_0\rangle},$$

$$F_\lambda(t) \leq ||h_\lambda(t)||^2 \equiv F_\lambda^{(c)}(t),$$

channel QFI *optimized over probe states* *maximal QFI*

Three types of Fisher informations

		probe state	measurement
Fisher information	$\mathcal{I}_\lambda$	specific	specific
QFI	F_λ	specific	optimized
channel QFI	$F_\lambda^{(c)}$	optimized	optimized

Seminorm of local generator

$$h_\lambda(t) = \int_0^t U_\lambda^\dagger(0 \to s) \frac{\partial H_\lambda(s)}{\partial \lambda} U_\lambda(0 \to s) ds.$$

triangle inequality

$$||\hat{A} + \hat{B}|| \leq ||\hat{A}|| + ||\hat{B}||$$

$$||h_\lambda(t)|| \leq \int_0^t \left|\left| U_\lambda^\dagger(0 \to s) \frac{\partial H_\lambda(s)}{\partial \lambda} U_\lambda(0 \to s) \right|\right| ds = \int_0^t \left|\left| \frac{\partial H_\lambda(s)}{\partial \lambda} \right|\right| ds,$$

$$I_g^{(Q)} \leq \left[\int_0^T (\mu_{\max}(t) - \mu_{\max}(t)) \mathrm{d}t \right]^2.$$

S. Pang and A. N. Jordan, Optimal adaptive control for quantum metrology with time-dependent hamiltonians, Nat. Commun. **8**, 1 (2017).

QFI inequality chain

QFI bound

$$F_\lambda^{(c)}(t) \leq \left[\int_0^t \left\|\frac{\partial H_\lambda(s)}{\partial \lambda}\right\| ds\right]^2 .$$

Inequality chain

$$\mathcal{I}_\lambda \leq F_\lambda \leq F_\lambda^{(c)} \leq \left[\int_0^t \left\|\frac{\partial H_\lambda(s)}{\partial \lambda}\right\| ds\right]^2 .$$

Sensitivity limits

$$\delta\lambda \geq \frac{1}{\sqrt{\nu}\int_0^t \left\| \frac{\partial H_\lambda(s)}{\partial\lambda} \right\| ds}.$$

spectral width

QFI flow

$$(\partial F_\lambda / \partial t) = 8\mathrm{Cov}[(\partial h_\lambda / \partial t), h_\lambda]|$$

$$\begin{aligned}\partial_x \mathrm{Var}(A) &= \partial_x \mathrm{Cov}(A, A) \\ &= 2\mathrm{Cov}(\partial_x A, A)\end{aligned}$$

$$\mathrm{Cov}[\hat{A}, \hat{B}]\Big|_{|\Psi\rangle} = \frac{1}{2}\langle\Psi|\hat{A}\hat{B} + \hat{B}\hat{A}|\Psi\rangle - \langle\Psi|\hat{A}|\Psi\rangle\langle\Psi|\hat{B}|\Psi\rangle.$$

QFI flow bound

$$\left| \mathrm{Cov}\left[\frac{\partial h_\lambda}{\partial t}, h_\lambda(t)\right]\Big|_{|\Psi_0\rangle} \right| \leq \sqrt{\mathrm{Var}\left[U_\lambda^\dagger \frac{\partial H_\lambda}{\partial \lambda} U_\lambda\right]\Big|_{|\Psi_0\rangle}} \, \frac{F_\lambda^{1/2}(t)}{2}$$
$$\leq \frac{\left\| \frac{\partial H_\lambda}{\partial \lambda} \right\|}{2} \frac{F_\lambda^{1/2}(t)}{2}. \quad (3)$$

$$F_\lambda(t) = 4(\langle \Psi_0 | h_\lambda^2(t) | \Psi_0 \rangle - |\langle \Psi_0 | h_\lambda(t) | \Psi_0 \rangle|^2)$$
$$\equiv 4\mathrm{Var}[h_\lambda(t)]|_{|\Psi_0\rangle},$$

$$\mathrm{Var}[\hat{A}]|_{|\Psi\rangle} \leq ||\hat{A}||^2/4.$$

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \;\; \geq \;\; \mathrm{Cov}^2(\mathrm{A}, \mathrm{B}) + \frac{|\langle [\mathrm{A}, \mathrm{B}] \rangle|^2}{4}$$

$$\left| \mathrm{Cov}(\hat{A}, \hat{B}) \right| \leq \sqrt{\mathrm{Var}(\hat{A})\mathrm{Var}(\hat{B})}.$$

QFI flow bound

$$\left|\frac{\partial F_\lambda^{1/2}(t)}{\partial t}\right| \leq \left\|\frac{\partial H_\lambda(t)}{\partial \lambda}\right\|.$$

QFI flow

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Quantum Fisher information flow and non-Markovian processes of open systems

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We establish an information-theoretic approach for quantitatively characterizing the non-Markovianity of open quantum processes. Here, the quantum Fisher information (QFI) flow provides a measure to statistically distinguish Markovian and non-Markovian processes. A basic relation between the QFI flow and non-Markovianity is unveiled for quantum dynamics of open systems. For a class of time-local master equations, the exactly analytic solution shows that for each fixed time the QFI flow is decomposed into additive subflows according to different dissipative channels.

Here we use the QFI to characterize the non-Markovianity of the open quantum system by introducing the QFI flow, which is defined as the change rate $\mathcal{I} := \partial\mathcal{F}/\partial t$ of the QFI. As a central result in this paper, a proposition about the decomposition of the QFI flow is given as follows.

Proposition. For an open quantum system described by the time local master equation (1), the QFI flow $\mathcal{I} = \sum_i \mathcal{I}_i$ is explicitly written as a sum of subflows $\mathcal{I}_i = \gamma_i \mathcal{J}_i$ with

$$\mathcal{J}_i := -\mathrm{Tr}\{\rho[L, A_i]^\dagger [L, A_i]\} \leqslant 0. \quad (6)$$

SCI引用
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From non-unitary to unitary dynamics

Dynamical process, reduced dynamics

$$R_S : \rho_S(0) \to \rho_S(t),$$

Mapped unitary dynamics

enlarge the quantum system

$$\mathcal{M}(U_{S,E}) \to R_S.$$

Hermitian Hamiltonian

$$U_{S,E} \to \tilde{H}_{\text{tot}},$$

may be complicated and unknown

Perturbation contains the parameter

$$\tilde{H}_{\text{tot}} + H_1(\lambda, t),$$

1. *Hermitian*
2. *only couples to system*

Ultimate sensitivity bound

$$\hat{H}_\lambda(t) = \hat{H}_S(\lambda, t) + \hat{H}_E(t) + \hat{H}_{SE}(t),$$

system environment interaction

$$\hat{H}_S(\lambda, t) = H_1(\lambda, t) + H_0(t),$$

the parameter only couples to the system described by a Hermitian Hamiltonian

$$\delta\lambda \geq \frac{1}{\sqrt{\nu}\int_0^t \left|\left|\frac{\partial H_\lambda(s)}{\partial\lambda}\right|\right| ds}.$$

$$\frac{\partial H_\lambda}{\partial \lambda} = \frac{\partial H_1}{\partial \lambda}$$

ultimate sensitivity bound

the ultimate sensitivity cannot be improved by:

1. *coupling to the environment*
2. *adding the auxiliary Hamiltonian*

The non-Hermitian quantum sensor will not outperform its Hermitian counterpart in the ultimate sensitivity.

W. Ding, X. Wang, and S. Chen,, PRL 131, 160801 (2023)(引用13).

Non-Hermitian Hamiltonian

$$\hat{H}_s = \mathcal{E}_\lambda \begin{pmatrix} 0 & \delta_\lambda^{-1} \\ \delta_\lambda & 0 \end{pmatrix},$$

to estimate λ

Simulated by the dilated two-qubit system

$$\hat{H}_{\text{tot}} = b\hat{I}^{(a)} \otimes \hat{\sigma}_x^{(s)} - c\hat{\sigma}_y^{(a)} \otimes \hat{\sigma}_y^{(s)} + \lambda \hat{I}^{(a)} \otimes \hat{\sigma}_x^{(s)},$$

The coefficients $b = 4\omega\varepsilon(1+\varepsilon)/(1+2\varepsilon)$ and $c = 2\omega\sqrt{\varepsilon(1+\varepsilon)}/(1+2\varepsilon)$,

$$|\Psi_0\rangle = \left(\sqrt{\frac{1+\varepsilon}{1+2\varepsilon}}|0\rangle_a + \sqrt{\frac{\varepsilon}{1+2\varepsilon}}|1\rangle_a\right) \otimes |0\rangle_s.$$

选择合适的初态，并对A进行后选择测量来进行模拟。由于A的Sigma_y守恒，其本质上等同一个比特。

$\mathcal{E}_\lambda = \sqrt{(b+\lambda)^2 + c^2}$ and $\delta_\lambda = (\lambda + 2\varepsilon\omega)/\mathcal{E}_\lambda$.

非厄米动力学

$$|\psi\rangle_s = e^{-i\hat{H}_s t}|0\rangle_s = \begin{pmatrix} \cos(\mathcal{E}_\lambda t) \\ -i\delta_\lambda \sin(\mathcal{E}_\lambda t) \end{pmatrix},$$

not normalized

normalized population

$$S(\lambda, t) = \frac{1}{1 + \delta_\lambda^2 \tan^2(\mathcal{E}_\lambda t)}.$$

$$\chi_s = \frac{\partial S}{\partial \lambda}$$

divergence of the susceptibility

when $\varepsilon \to 0$

eigenstate coalescence

幺正动力学

$$|\Psi(\tau)\rangle = e^{-i\hat{H}_{\mathrm{tot}}\tau}|\Psi_0\rangle,$$

$$\hat{H}_{\mathrm{tot}} = b\hat{I}^{(a)} \otimes \hat{\sigma}_x^{(s)} - c\hat{\sigma}_y^{(a)} \otimes \hat{\sigma}_y^{(s)} + \lambda\hat{I}^{(a)} \otimes \hat{\sigma}_x^{(s)},$$

$$|\Psi_0\rangle = \left(\sqrt{\frac{1+\varepsilon}{1+2\varepsilon}}|0\rangle_a + \sqrt{\frac{\varepsilon}{1+2\varepsilon}}|1\rangle_a\right) \otimes |0\rangle_s.$$

Sensitivity

Probability in state $|0_a\rangle \otimes |0_s\rangle$

$$P_1 = \frac{1+\varepsilon}{1+2\varepsilon}\cos^2\left[t\sqrt{\lambda^2 + \frac{8\varepsilon(1+\varepsilon)\lambda\omega}{1+2\varepsilon} + 4\varepsilon(1+\varepsilon)\omega^2}\right].$$

uncertainty due to projection measurement
实际测量都是有限次

$$\mathrm{Var}[\hat{P}_1] = \frac{P_1(1-P_1)}{\nu},$$

spin projection noise
binomial distribution

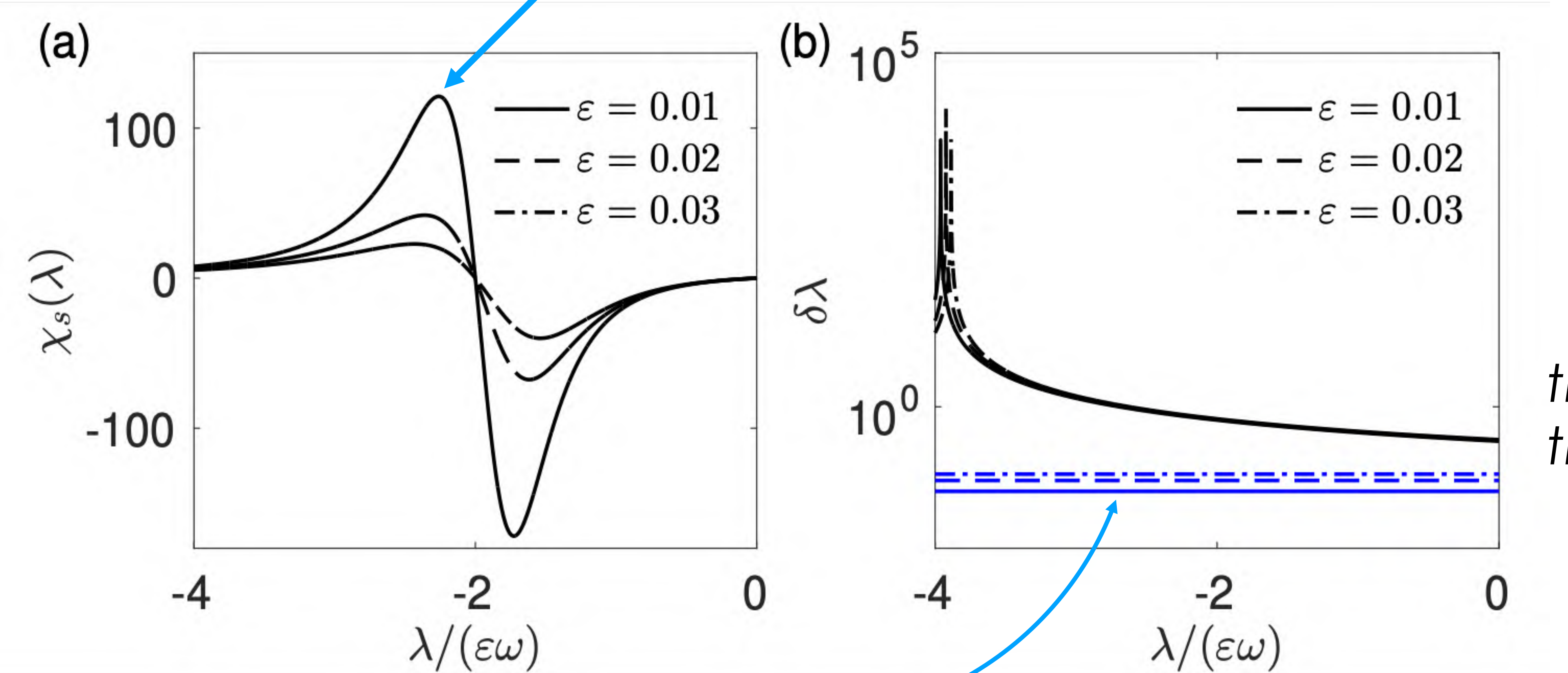
sensitivity (error propagation formula)

$$\delta\lambda = \frac{\sqrt{\mathrm{Var}[\hat{P}_1]}}{\left|\frac{\partial P_1}{\partial \lambda}\right|}.$$

信噪比

No divergence in the sensitivity

the measurement point



the smaller, the better.

Hermitian counterpart

$$\hat{V} = \lambda \hat{I}^{(a)} \otimes \hat{\sigma}_x^{(s)}$$

$$\delta\lambda \geq \frac{1}{\sqrt{\nu\tau}||\hat{\sigma}_x||} = \frac{1}{2\sqrt{\nu\tau}}.$$

EP based sensor using single trapped ion

Periodically driven $\mathcal{PT}$-symmetric *non-Hermitian Floquet system*

$$\hat{H}_{\mathcal{PT}} = J[1+\cos(\omega t)]\hat{\sigma}_x + i\Gamma\hat{\sigma}_z,$$

Hamiltonian implemented *in the experiment*

$$\hat{H}'_{\mathcal{PT}} = \hat{H}_{\mathcal{PT}} - i\Gamma\hat{I},$$

Perturbation

$$\hat{H}_\delta = \frac{\delta}{2}\cos(\omega_\delta t)(\hat{I} - \hat{\sigma}_z),$$ *to estimate ω_δ*

L. Ding, K. Shi, Q. Zhang, D. Shen, X. Zhang, and W. Zhang, Experimental determination of $\mathcal{PT}$-symmetric exceptional points in a single trapped ion, Phys. Rev. Lett. **126**, 083604 (2021).

Response energy

$$P_J(T) - P_\Gamma(T) = \sin^2(\mathcal{E}_{\text{res}} T),$$

measurable quantities

spin population measurement

$$P_J(T) = \left| \langle\uparrow| \mathcal{T} e^{-i\int_0^T [\hat{H}_{\mathcal{PT}}(t) + \hat{H}_\delta(t)]dt} |\downarrow\rangle \right|^2,$$

$$P_\Gamma(T) = \left| \frac{\langle\uparrow| - \langle\downarrow|}{\sqrt{2}} \mathcal{T} e^{-i\int_0^T [\hat{H}_{\mathcal{PT}}(t) + \hat{H}_\delta(t)]dt} \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} \right|^2.$$

Propagation of uncertainty

uncertainty in P_J and P_Γ

$$C_0 \equiv e^{2\Gamma T}$$

$$\mathrm{Var}[\hat{P}_i] = \frac{P_i(C_0 - P_i)}{\nu}, \text{with } i = J, \Gamma,$$

due to H'_{PT} in experiment

propagation of uncertainty

$$\mathrm{Var}[f(X)] \approx (f'(\mathrm{E}[X]))^2 \mathrm{Var}[X],$$

valid when $Var[X]$ is small enough

$$\begin{aligned}\mathrm{Var}[\hat{\mathcal{E}}_{\mathrm{res}}] &= \frac{1}{4T^2} \frac{\mathrm{Var}[\hat{P}_J] + \mathrm{Var}[\hat{P}_\Gamma]}{(P_J - P_\Gamma)(1 - P_J + P_\Gamma)} \\ &= \frac{1}{4\nu T^2} \frac{C_0(P_J + P_\Gamma) - (P_J^2 + P_\Gamma^2)}{(P_J - P_\Gamma)(1 - P_J + P_\Gamma)},\end{aligned}$$

Overall sensitivity

$$\delta\omega_\delta = \frac{\sqrt{\mathrm{Var}[\hat{\mathcal{E}}_{\mathrm{res}}]}}{|\frac{\partial \mathcal{E}_{\mathrm{res}}}{\partial \omega_\delta}|}.$$

Hermitian counterpart

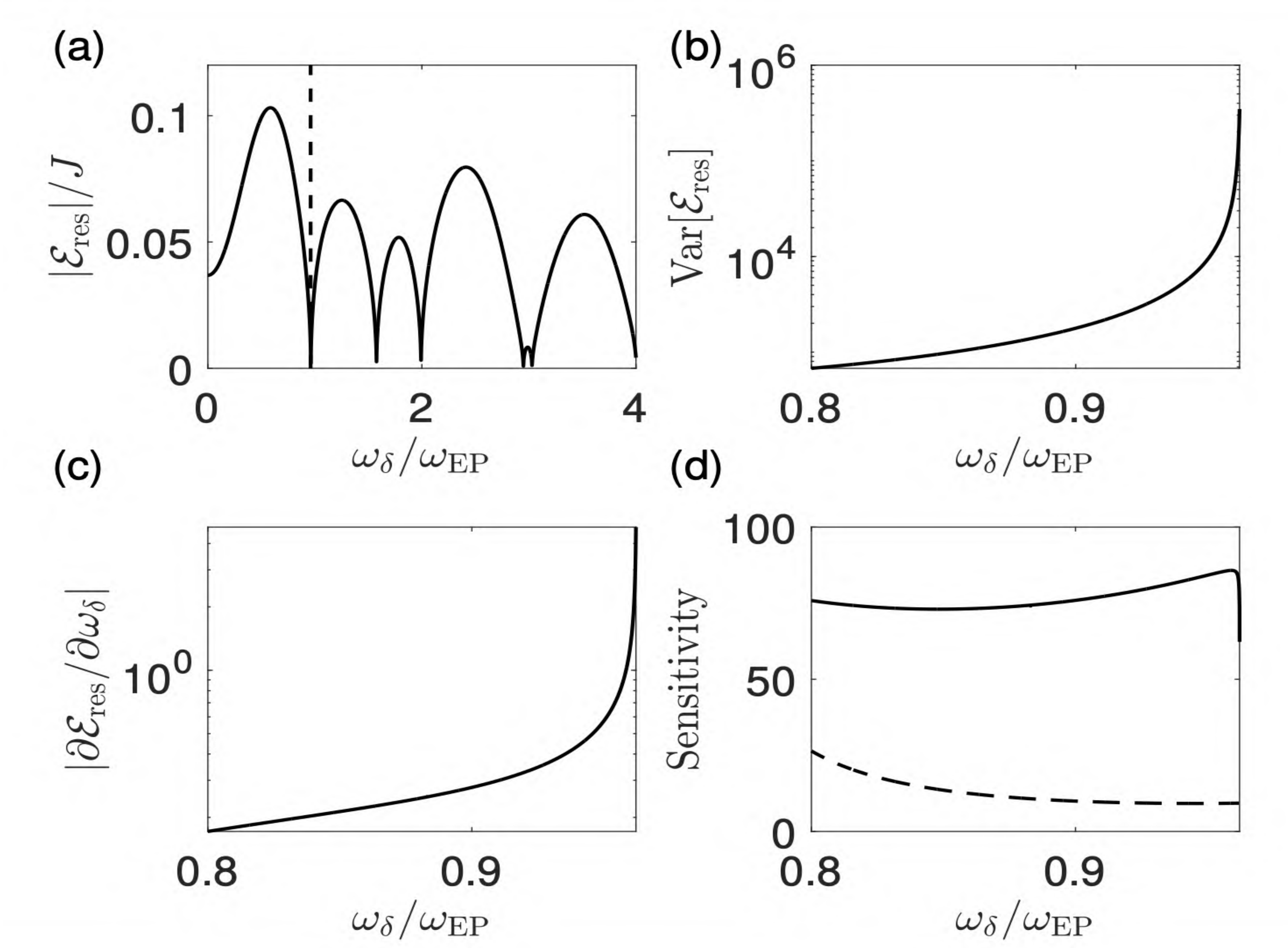
$$\hat{H}_\delta = \frac{\delta}{2}\cos(\omega_\delta t)(\hat{I} - \hat{\sigma}_z),$$

$$\delta\omega_\delta \geq \frac{\omega_\delta^4}{\sqrt{\nu}\delta^2[\sin(\omega_\delta T) - \omega_\delta T\cos(\omega_\delta T)]^2}.$$

Sensitivity

sharp dips near the EP

divergence in variance



Hermitian counterpart

divergence in susceptibility

no divergence in sensitivity

Conclusions

1、We have unveiled the fundamental sensitivity limit for non-Hermitian sensors in the context of open quantum systems. Our results indicate clearly that non-Hermitian sensors do not outperform their Hermitian counterparts.

2、Although our work demonstrates that coupling to the environment cannot improve the ultimate sensitivity, when the probe state or the measurement protocol is restricted, adding appropriate auxiliary Hamiltonian may be helpful for approaching the ultimate sensitivity bound.

谢谢各位老师同学！